Probability:

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity. An **experiment** is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a chance experiment.
Example of an experiment: Flipping one fair coin twice.

A result of an experiment is called an **outcome**. The **sample space** of an experiment is the set of all possible outcomes. Three ways to represent a sample space are: to list the possible outcomes, to create a tree diagram, or to create a Venn diagram. The uppercase letter *S* is used to denote the sample space.
Example: if you flip one fair coin, $S = \{H, T\}$ where H = heads and T = tails are the outcomes.

An **event** is any combination of outcomes. Upper case letters like *A* and *B* represent events. For example, if the experiment is to flip one fair coin, event *A* might be getting at most one head.
Example: The probability of an event *A* is probability of getting at most one head. It is also written as $P(A)$.

The **probability** of any outcome is the **long-term relative frequency** of that outcome. **Probabilities are between zero and one, inclusive** (that is, 0 ≤≤ probability of an event ≤≤ 1).

- $P(A) = 0$ means the event *A* can never happen.
- $P(A) = 1$ means the event *A* always happens.
- $P(A) = 0.5$ means the event *A* is equally likely to occur or not to occur.

Example: If you flip one fair coin repeatedly (from 20 to 2,000 to 20,000 times) the relative frequency of heads approaches 0.5 (the probability of heads).

Equally likely means that each outcome of an experiment occurs with equal probability.

Example:

1. If you toss a **fair**, six-sided die, each face (1, 2, 3, 4, 5, or 6) is as likely to occur as any other face.
2. If you toss a fair coin, a Head (*H*) and a Tail (*T*) are equally likely to occur.
3. If you randomly guess the answer to a true/false question on an exam, you are equally likely to select a correct answer or an incorrect answer.

To calculate the probability of an event *A* when all outcomes in the sample space are equally likely, count the number of outcomes for event *A* and divide by the total number of outcomes in the sample space.

For examples:

- If you toss two coins, you will see four possible outcomes. These 4 outcomes will form a sample space. Therefore, the sample space is {*HH*, *TH*, *HT*, *TT*} where *T* = tails and *H* = heads.
  If event *A* = getting one head, then there are two outcomes that meet this condition {*HT*, *TH*}.

The probability of event A, P(A) = number of outcome with only one head / total of possible outcomes = 2/4 = 0.5.

- Suppose you roll one fair six-sided die, with the numbers {1, 2, 3, 4, 5, 6} on its faces. Let event *E* = rolling a number that is at least five. There are two outcomes {5, 6}.

P(E) = number of outcome that rolling a number that is at least five /total of possible outcomes =2/6 as the number of repetitions grows larger and larger.

OR Event

An outcome is in the event *A* OR *B* if the outcome is in *A* or is in *B* or is in both *A* and *B*. For example, let *A* = {1, 2, 3, 4, 5} and *B* = {4, 5, 6, 7, 8}.

A OR *B* = {1, 2, 3, 4, 5, 6, 7, 8}.

AND Event

An outcome is in the event *A* AND *B* if the outcome is in both *A* and *B* at the same time. For example, let *A* = {1, 2, 3, 4, 5} and B = {4, 5, 6, 7, 8}, respectively. Then

$$A \text{ AND } B = \{4, 5\}.$$

Complimentary Event

The **complement** of event *A* is denoted *A′* (read "*A* prime"). *A′* consists of all outcomes that are **NOT** in *A*.

$$P(A) + P(A') = 1.$$

For example:

We have sample space *S* = {1, 2, 3, 4, 5, 6}.
If event *A* = {1, 2, 3, 4}, then event A'= {5, 6}.
P(A) = 4/6 and P(A') = 2/6
P(A) + P(A') = (4/6)+(2/6) =1

Conditional Probability of Event

The **conditional probability** of *A* given *B* is written *P*(*A*|*B*). *P*(*A*|*B*) is the probability that event *A* will occur given that the event *B* has already occurred. **A conditional reduces the sample space.** We calculate the probability of *A* from the reduced sample space *B*. The formula to calculate *P*(*A*|*B*) is

P(A|B)=P(A AND B) / P(B) ; where *P*(*B*) is greater than zero.

For example:

Suppose we toss one fair, six-sided die. The sample space *S* = {1, 2, 3, 4, 5, 6}.
Let event *A* = face is 2 or 3 and *B* = event that face is even.
Event A = {2, 3}, Event B = {2, 4, 6}.

To calculate *P*(*A*|*B*), we count the number of outcomes 2 or 3 in the sample space *B* = {2, 4, 6}. Then we divide that by the number of outcomes *B* (rather than *S*).

We get the same result by using the formula. Remember that *S* has six outcomes.

A and B = {2} (as 2 appears in both event A and event B.)

P(A|B)=P(A AND B) / P(B) = (the number of outcomes that are in both event A and event B / total outcomes) / (the number of outcomes in event B / total outcomes) =(1/ 6) / (3/6) =1/3

Types of Probability

There are three major types of probabilities:

- Theoretical Probability
- Experimental Probability
- Axiomatic Probability

- **Theoretical Probability**

It is based on the possible chances of something to happen. The theoretical probability is mainly based on the reasoning behind probability. For example, if a coin is tossed, the theoretical probability of getting a head will be ½.

- **Experimental Probability**

It is based on the basis of the observations of an experiment. The experimental probability can be calculated based on the number of possible outcomes by the total number of trials. For example, if a coin is tossed 10 times and heads is recorded 6 times then, the experimental probability for heads is 6/10 or, 3/5.

- **Axiomatic Probability**

In axiomatic probability, a set of rules or axioms are set which applies to all types. With the axiomatic approach to probability, the chances of occurrence or non-occurrence of the events can be quantified.

Rules of Probability

Often, we want to compute the probability of an event from the known probabilities of other events. This lesson covers some important rules that simplify those computations.

Definitions and Notation

Before discussing the rules of probability, we state the following definitions:

- Two events are **mutually exclusive** or **disjoint** if they cannot occur at the same time.
- The probability that Event A occurs, given that Event B has occurred, is called a **conditional probability**. The conditional probability of Event A, given Event B, is denoted by the symbol P(A|B).
- The **complement** of an event is the event not occurring. The probability that Event A will not occur is denoted by P(A').
- The probability that Events A and B *both* occur is the probability of the **intersection** of A and B. The probability of the intersection of Events A and B is denoted by P(A ∩ B). If Events A and B are mutually exclusive, P(A ∩ B) = 0.
- The probability that Events A or B occur is the probability of the **union** of A and B. The probability of the union of Events A and B is denoted by P(A ∪ B) .
- If the occurrence of Event A changes the probability of Event B, then Events A and B are **dependent**. On the other hand, if the occurrence of Event A does not change the probability of Event B, then Events A and B are **independent**.

Rule of Subtraction

In a previous lesson, we learned two important properties of probability:

- The probability of an event ranges from 0 to 1.
- The sum of probabilities of all possible events equals 1.

The rule of subtraction follows directly from these properties.

Rule of Subtraction. The probability that event A will occur is equal to 1 minus the probability that event A will not occur.

$$P(A) = 1 - P(A')$$

Suppose, for example, the probability that Bill will graduate from college is 0.80. What is the probability that Bill will not graduate from college? Based on the rule of subtraction, the probability that Bill will not graduate is 1.00 - 0.80 or 0.20.

Rule of Multiplication

The rule of multiplication applies to the situation when we want to know the probability of the intersection of two events; that is, we want to know the probability that two events (Event A and Event B) both occur.

Rule of Multiplication The probability that Events A and B both occur is equal to the probability that Event A occurs times the probability that Event B occurs, given that A has occurred.

$$P(A \cap B) = P(A)\ P(B|A)$$

Example

An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *without replacement* from the urn. What is the probability that both of the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, $P(A) = 4/10$.
- After the first selection, there are 9 marbles in the urn, 3 of which are black. Therefore, $P(B|A) = 3/9$.

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A)\ P(B|A)$$
$$P(A \cap B) = (4/10) * (3/9) = 12/90 = 2/15 = 0.133$$

Rule of Addition

The rule of addition applies to the following situation. We have two events, and we want to know the probability that either event occurs.

Rule of Addition The probability that Event A or Event B occurs is equal to the probability that Event A occurs plus the probability that Event B occurs minus the probability that both Events A and B occur.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: Invoking the fact that $P(A \cap B) = P(A)P(B \mid A)$, the Addition Rule can also be expressed as:

$$P(A \cup B) = P(A) + P(B) - P(A)P(B \mid A)$$

Example

A student goes to the library. The probability that she checks out (a) a work of fiction is 0.40, (b) a work of non-fiction is 0.30, and (c) both fiction and non-fiction is 0.20. What is the probability that the student checks out a work of fiction, non-fiction, or both?

Solution: Let F = the event that the student checks out fiction; and let N = the event that the student checks out non-fiction. Then, based on the rule of addition:

$$P(F \cup N) = P(F) + P(N) - P(F \cap N)$$
$$P(F \cup N) = 0.40 + 0.30 - 0.20 = 0.50$$

Conditional Probability

The ***conditional probability*** of an event *B* is the probability that the event will occur given the knowledge that an event *A* has already occurred. This probability is written *P(B|A)*, notation for the *probability of B given A*. In the case where events *A* and *B* are *independent* (where event *A* has no effect on the probability of event *B*), the conditional probability of event *B* given event *A* is simply the probability of event *B*, that is *P(B)*.

If events *A* and *B* are not independent, then the probability of the *intersection of A and B* (the probability that both events occur) is defined by
P(A and B) = P(A)P(B|A).

From this definition, the conditional probability *P(B|A)* is easily obtained by dividing by *P(A)*:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Examples

In a card game, suppose a player needs to draw two cards of the same suit in order to win. Of the 52 cards, there are 13 cards in each suit. Suppose first the player draws a heart. Now the player wishes to draw a second heart. Since one heart has already been chosen, there are now 12 hearts remaining in a deck of 51 cards. So the conditional probability *P(Draw second heart|First card a heart)* = 12/51.

Another important method for calculating conditional probabilities is given by ***Bayes's formula***. The formula is based on the expression

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Where:

- P(A|B) – the probability of event A occurring, given event B has occurred
- P(B|A) – the probability of event B occurring, given event A has occurred
- P(A) – the probability of event A
- P(B) – the probability of event B

Bayes' Theorem

You might be interested in finding out a patient's probability of having liver disease if they are an alcoholic. "Being an alcoholic" is the **test** (kind of like a litmus test) for liver disease.

- **A** could mean the event "Patient has liver disease." Past data tells you that 10% of patients entering your clinic have liver disease. P(A) = 0.10.
- **B** could mean the litmus test that "Patient is an alcoholic." Five percent of the clinic's patients are alcoholics. P(B) = 0.05.
- You might also know that among those patients diagnosed with liver disease, 7% are alcoholics. This is your **B|A:** the probability that a patient is alcoholic, given that they have liver disease, is 7%.

According to Bayes' theorem:
P(A|B) = (0.07 * 0.1)/0.05 = 0.14
In other words, if the patient is an alcoholic, their chances of having liver disease is 0.14 (14%). This is a large increase from the 10% suggested by past data. But it's still unlikely that any particular patient has liver disease.

Probability Distributions:

Binomial Distribution:

The binomial is a type of distribution that **has two possible outcomes** (the prefix "bi" means two, or twice). For example, a coin toss has only two possible outcomes: heads or tails and taking a test could have two possible outcomes: pass or fail. A Binomial Distribution shows either (S)uccess or (F)ailure.

The binomial distribution formula is for any random variable X, given by; **P(x:n,p) = ^nCx * p^x $(1-p)^{n-x}$ Or P(x:n,p) = ^nCx * p^x $(q)^{n-x}$**, where, n is the number of experiments, p is probability of success in a single experiment, q is probability of failure in a single experiment (= 1 – p) and takes values as 0, 1, 2, 3, 4, ...

where,
n= the number of trials
x = the number of successes desired
p = probability of getting a success in one trial
q = (1-p) = the probability of getting a failure in one trial
and

$^nCx = \frac{n!}{x!(n-x)!}$

Poisson Distribution:

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. The Poisson distribution can also be used for the number of events in other specified interval types such as distance, area or volume.

$$F(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Where:

- *e* is Euler's number (*e* = 2.71828...)
- *x* is the number of occurrences
- λ is equal to the expected value

Normal distribution, also known as the Gaussian distribution, is a **probability distribution that is symmetric about the mean**, showing that data near the mean are more frequent in occurrence than data far from the mean. In graph form, normal distribution will appear as a bell curve.